

Capturing Four Stylized Facts of Financial Time Series in GARCH and Stochastic Volatility Models

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Abstract: In this paper, four stylized empirical features about the log returns of financial data sets from the point of view in statistical analysis will be described, which are fat-tails, the gain/loss asymmetry, the absence of autocorrelation and volatility clustering. Those properties will be illustrated with three assets from different financial markets. Because these features cannot be captured by any single model, it is necessary to discuss several models which consider kurtosis-autocorrelation combinations and volatilities. The purpose of this paper is to compare three theoretical models, GARCH, ARSV and EGARCH, figuring out how well they can reproduce those stylized features, and examine them through forecasting, which is helpful for modifying and establishing more proper empirical models in order to perform reliable analysis. Finally, it will be pointed out that none of the models dominates the others when comparing their ability to capture four features. However, ARSV can better reproduce the original data set and it can generate smaller value of autocorrelations with larger range of values of parameters.

1. Introduction

The goal of this paper is to summarize basic knowledge about three models, GRACH, EGARCH and ARSV, and compare them. The general statistical properties and widely known empirical qualitative truths about asset's returns are called stylized features or stylized facts. Four features will be discussed in Section 2, which will be considered for modelling part. The focus of this paper is to compare the statistical modelling of financial data series to find out how well those models can reproduce the stylized features. Section 3 will summarize three theoretical models and compare them theoretically. In section 4, forecasting will be applied and comparisons among these models in empirical perspective will be discussed. Section 5 contains conclusions. In this article, the daily closing prices of MSFT, EUR/USD exchange rate and Crude Oil Contract from January 1, 2000 to January 31, 2020 are used for analysis and discussion, which are transformed into a log-return series to make the data comparable and stable.

2. Stylized Features of Asset Returns

2.1. Fat Tails

The first stylized feature is fat tails, which means that the probability of observing extreme values is higher than corresponding normal distribution with same mean and variance. Kurtosis is often used to quantify the peakedness and measures the tails of the distribution. A distribution with kurtosis value higher than 3 is defined as leptokurtic. See Table 1, the distributions of all three assets are leptokurtic.

We can visualize fat tails using histogram and density plot. The peaks of all histogram of daily log returns of three financial data are higher and the tails are fatter at both side, which suggests that all three data sets are prone to outlier and daily returns of the market assets tend to have excess kurtosis.

2.2. Asymmetry

The second stylized feature is gain/loss asymmetry. The skewness measures the symmetry of distribution with respect to the mean. All symmetric distribution has skewness value equal to zero. The descriptive statistics are summarized in Table 1. The negative skewness value implies that MSFT and crude oil data are spread out toward left and the right tail is fatter than the left one, while the distribution of EUR/USD spread out toward right and its distribution is concentrated on the left.

Table 1: Descriptive statistics of daily log return for three financial assets.

Asset	mean	variance	skewness	kurtosis
MSFT	0.03025333	3.595063	-0.1311608	12.80778
EUR/USD	0.001482144	0.3699958	0.05229373	4.76669
Crude Oil	0.01372379	5.528808	-0.09026126	7.145502

2.3. Lack of Autocorrelation

The third stylized feature about the dependence of log-return distribution is the absence of autocorrelation, which measures the correlation of returns between a given day and lagged days from the same data series. From the ACF plot of MSFT, the autocorrelation function for MSFT decays to zero after a lag, which implies that the price movements do not have significant linear autocorrelation. This feature suggests that traditional method, such as autocovariance analysis, ARMA modelling and Fourier analysis, cannot distinguish white noise from the actual returns [2].

2.4. Volatility Clustering

The fourth stylized feature is the volatility clustering, which implies that prices have time-varying volatility. The signs of successive price movements are independent, but the magnitude of absolute or squared returns have significant positive autocorrelation and its decay slowly, which suggests that we can predict changes of returns based on previous performance by volatility modeling. Large price changes tend to cluster in time and the volatilities are constant. With this feature, the random walk modelling is not a good fit because it cannot accommodate for time-varying volatility, or Gaussian formulation with heavy-tailed increments.

3. Theoretical Models

3.1. GARCH Model

Bollerslev (1986) introduced standard GARCH (p, q) model, which represents the dynamic evolution of the volatility of financial returns. GARCH (1, 1) will be discussed in details for its widely application and simplicity, which is given by:

$$r_t = \mu + \varepsilon_t \quad (1)$$

$$\varepsilon_t = \eta_t \sigma_t \quad (2)$$

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 + b_1 \sigma_{t-1}^2 \quad (3)$$

where $\{\eta_t\}$ is independently identically distributed random variables with mean equal to zero.

For the first stylized fact that GARCH model catches, the excess kurtosis, we need to consider the fourth moment which exists if and only if $a_1^2 E(\eta_t^4) + 2a_1 b_1 E(\eta_t^2) + b_1^2$ is less than one[5]. The kurtosis of η_t equals to

$$k_4 = \frac{E(\eta_t^4) / E(\eta_t^2)^2 (1 - (a_1 E(\eta_t^2) + b_1)^2)}{1 - (a_1^2 E(\eta_t^4) + 2a_1 b_1 E(\eta_t^2) + b_1^2)} \quad (4)$$

Assuming normality for the distribution of $\{\eta_t\}$, k_4 is larger than three, which is consist with the property of excess kurtosis.

For the third and fourth features, the autocorrelation function of $\{\eta_t^2\}$ is discussed, which is defined as:

$$\rho_n = (a_1 E(\eta_t^2) + b_1)^{n-1} \frac{a_1 E(\eta_t^2) (1 - b_1^2 - a_1 b_1 E(\eta_t^2))}{1 - b_1^2 - 2a_1 b_1 E(\eta_t^2)}, n \geq 1 \quad (5)$$

The decay of autocorrelation is shown by the exponential part, $(a_1 E(\eta_t^2) + b_1)^{n-1}$. $a_1 E(\eta_t^2) + b_1$ measures the persistence of volatility clustering of squared returns and $a_1 E(\eta_t^2)$ measures of the dependence of squared returns with a given persistence. Autocorrelation increases with the persistence when a_1 is given.

Given a constant persistence, both the kurtosis and autocorrelation increase with a_1 . The function of the relationship between kurtosis and autocorrelation under the normality of $\{\eta_t\}$ can be derived by combining each individual function:

$$\rho_n = (a_1 + b_1)^{n-1} \left(\frac{b_1 (1 - 3k_4^{-1})}{3 - 3k_4^{-1}} + a_1 \right), n \geq 1 \quad (6)$$

From the plot, the autocorrelation increases with the kurtosis when persistence is given and the first-order autocorrelation increases rapidly at first with kurtosis and persistence and then the increasement slows down.

To match the first stylized feature, heavy-tailed distribution can be applied to the original GARCH model, such as the student's t distribution and General Error Distribution (GED) [8][9]. However, it showed that the kurtosis coefficient and the student's t GARCH model will cause biased and inconsistent estimators of the degree of freedom. Feng, Shi (2017) compared four different GARCH

models, showing that GARCH model with tempered stable distribution consistently outperforms those with normal, student's t and GED distributions [11].

3.2. ARSV Model

ARSV (1) model was introduced by Taylor (1986) [12]. The volatility is transformed into logarithms and depends on unobserved latent variable. The model is defined as:

$$\begin{aligned} \varepsilon_t &= z_t \sigma_t, \{z_t\} \sim \text{nid}(0, 1) & (7) \\ \ln \sigma_t^2 &= \omega + \beta \ln \sigma_{t-1}^2 + \eta_t, \{\eta_t\} \sim \text{nid}(0, \sigma_\eta^2) & (8) \end{aligned}$$

where $\{\eta_t\}$ are independent normal distributed random variables with mean zero and variance σ_η^2 , error terms $\{z_t\}$ and $\{\eta_t\}$ are mutually independent.

From standard theory, all moments of $\ln \sigma_t^2$ exists if and only if $|\beta| < 1$. The kurtosis of ε_t is given by:

$$k_4 = k_4(z_t) \exp \{\sigma_\eta^2\} \quad (9)$$

where $\sigma_h^2 = \sigma_\eta^2 / (1 - \beta^2)$, which increases with σ_η^2 monotonically. When $\{z_t\}$ follows the normal distribution, k_4 is larger than three, which means ε_t is leptokurtic.

The second stylized feature can be captured by ARSV (1) when $\{z_t\}$ are correlated to $\{\eta_t\}$.

To analyze how ARSV (1) model match the fourth feature, the autocorrelation function of $\{|\varepsilon_t|^{2m}\}$ is considered:

$$\rho_n(m) = \frac{\exp(m^2 \sigma_h^2 \beta^n) - 1}{k_m \exp(m^2 \sigma_h^2) - 1}, n \geq 1 \quad (10)$$

where $k_m = E|z_t|^{4m} / (E|z_t|^{2m})^2$ [13]. The β represents a measurement of persistence in ARSV (1) model. Malmsten & Teräsvirta (2004) pointed out that the decay rate of squared observations is faster than exponential and stabilizes to β as the lag length increases [16]. Carnero, Pena & Ruiz (2004) showed that the approximated autocorrelations are larger than the true ones and the decay rate of approximated autocorrelations is smaller than that of true autocorrelations when the lags are small [17]. The relationship between kurtosis, persistence and autocorrelation of $\{|\varepsilon_t|^2\}$ is

$$\rho_n(1) = \frac{(k_4/k_4(z_t))^{\beta^n} - 1}{k_4 - 1}, n \geq 1 \quad (11)$$

Similar to GARCH (1, 1) model, the first stylized feature can also be captured by ASRV model as well. For example, the ARSV model with student's t distribution can be used to describe heavy tails and volatility dependencies [18]. When the number of degrees of freedom in student's t distribution decreases, the effect on first-order autocorrelation from persistence β can be neglected, and the autocorrelation decreases rapidly with the number of degrees [16].

3.3. EGARCH Model

EGARCH model, proposed by Nelson (1991), has direct relationship with the GARCH (1, 1) and ARSV (1) model. Like ARSV (1) model, the volatility term is defined using logarithm and its lags ensures that σ_t^2 is positive. The specification of volatility of EGARCH (1, 1) model is:

$$r_t = \mu + \varepsilon_t \quad (12)$$

$$\varepsilon_t = \sigma_t z_t \quad (13)$$

$$\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - E[|z_{t-1}|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2) \quad (14)$$

where μ is expected return, ε_t is a zero-mean white noise and z_t follows standard gaussian distribution [19].

The kurtosis of ε_t , assuming z_t follows the gaussian distribution, is given by:

$$k_4 = 3 \exp\left(\frac{(\alpha+\gamma)^2}{1-\beta^2}\right) \prod_{i=1}^{\infty} \frac{\Phi(2\beta^{i-1}(\alpha+\gamma)) + \exp(-8\beta^{2i-2}\alpha\gamma)\Phi(2\beta^{i-1}(\alpha-\gamma))}{(\Phi(\beta^{i-1}(\alpha+\gamma)) + \exp(-2\beta^{2i-2}\alpha\gamma)\Phi(\beta^{i-1}(\alpha-\gamma)))^2} \quad (15)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. It is obvious that k_4 is larger than three, which matches excess kurtosis of financial data series.

The parameter γ measures the asymmetry, the second feature, and the term $\alpha(|z_{t-1}| - E[|z_{t-1}|])$ represents a magnitude effect. Including parameter γ can reproduce financial data series with a larger range of values of kurtosis and autocorrelations. The error $\{z_t\}$ has all moments if and only if $|\beta| < 1$. When $\alpha = 0$, the moment expressions of two models, EGARCH (1, 1) and ARSV (1), are similar.

For measuring the third and fourth features, the autocorrelation of squared errors is derived and is given by:

$$\rho_n(1) = \frac{E(\varepsilon_{t-1}^2 \exp(\beta^{n-1}g))p_1 p_2 - p_3}{k_\varepsilon p_4 - p_3} \quad (16)$$

where $g = \alpha(|z_{t-1}| - E[|z_{t-1}|]) + \gamma z_{t-1}$, $p_1 = \prod_{i=1}^n E(\exp(\beta^{i-1}g))$, $p_2 = \prod_{i=1}^{\infty} E(\exp((1+\beta^n)\beta^{i-1}g))$, $p_3 = \prod_{i=1}^{\infty} (E(\exp(2\beta^{i-1}g)))^2$, $p_4 = \prod_{i=1}^{\infty} E(\exp(2\beta^{i-1}g))$ [19]. The value of autocorrelation decreases when β decreases. The decay rate of the autocorrelation depends on β as well as on α and γ . Also, the decay rate is faster than exponential at short lags and approaches β when the lag length increases [16].

3.4. Comparison of Three Models

First of all, the volatility terms in ARSV (1) and EGARCH (1, 1) model are transformed into logarithms, which is different from GARCH (1, 1) model. Second, EGARCH model overcomes some drawbacks of the GARCH model, which cannot match asymmetry of the returns. Also, there are restrictions for parameters of GARCH model to guarantee positivity of the conditional variance and it is difficult to measure the persistence of GARCH model [17].

Moreover, unlike EGARCH (1, 1) and GARCH (1, 1), ARSV (1) can describe financial data series with high kurtosis and low first-order autocorrelation of squared observations. Given same kurtosis and persistence, the value of autocorrelation of squared errors in ARSV (1) is lower than that in

EGARCH (1, 1) and GARCH (1, 1). Therefore, ARSV (1) can be a better fit for financial data series sometimes.

Furthermore, the relationships between kurtosis, persistence and first-order autocorrelation of GARCH (1, 1) and ARSV (1) with normal errors are compared, see Figure 1. In the plot, the first-order autocorrelation of ARSV (1) is increasing with persistence β while the first-order autocorrelation of GARCH (1, 1) is decreasing with persistence $a_1 + b_1$. The low first-order autocorrelation and high persistence can be satisfied at the same time for ARSV (1) model than GARCH (1, 1) model.

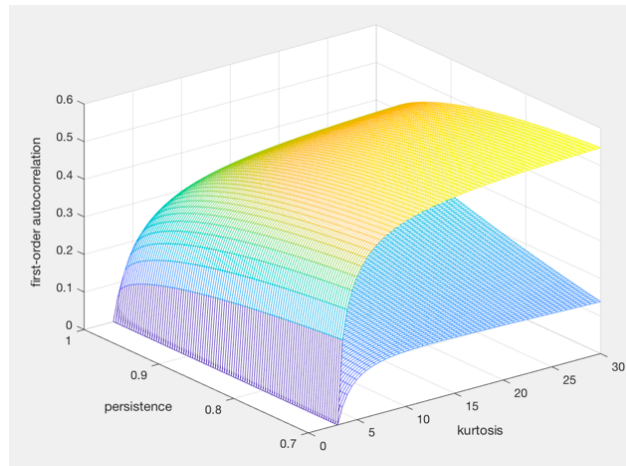


Figure 1: Relationship between kurtosis, persistence and first-order autocorrelation of GARCH (1,1) (up) and ARSV (1) (down) with normal errors.

4. Empirical Model Fitting and Forecasting

We will perform forecasting by fitting three models discussed above on returns of MSFT, EUR/USD exchange rate and Crude Oil. The result of fitting and forecasting of MSFT will be showed in details. Data from the first 15 years will be treated as training data and the rest will be treated as test data.

4.1. GARCH (1, 1)

From the ACF plot of squared standardized residuals of GARCH (1, 1) for MSFT, the residuals do not have autocorrelation. The plot about fitted series with two conditional SD superimposed of GARCH (1, 1) shows that our model explains most historical information of original financial data series.

Based on the analysis in section 2.1, the data has fat tails, which implies that we can fit data with GARCH (1, 1) model with student's t distribution. Comparing two QQ plots of residuals of standard GARCH and GARCH- t model, we know that the residuals from GARCH- t model satisfy normal distribution, which means that GARCH- t model can explain more data properties and represent the excess kurtosis and fat tails of the distribution of sample financial data series at the same time.

4.2. ARSV (1)

The default plot of the estimated ARSV (1) model on MSFT plots a summary of posterior density of the estimated volatility through its 5%, 50% and 95% quantiles, the Markov chain of three parameters

and prior and posterior density plots of the three parameters, which gives us a general visualized information of the fitted model. For predicting the log return of MSFT in 2015 to 2020, we quantify the uncertainties of predictions through quantiles (5%, 50%, 95%). See Figure 2.

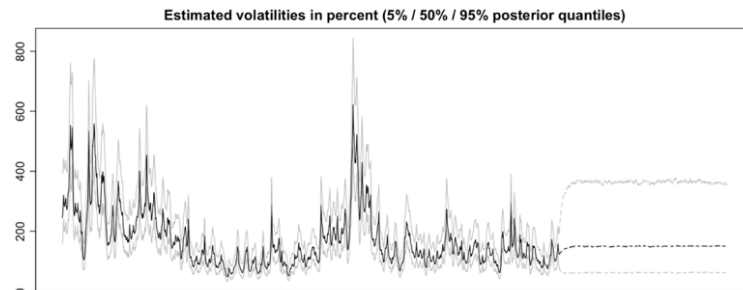


Figure 2: prediction of ARSV (1) for MSFT with 50% (black) and 5% / 95% quantiles (gray).

4.3. EGARCH (1, 1)

From ACF plot of squared standardized residuals and plot about fitted series with two conditional SD superimposed of EGARCH (1, 1) for MSFT, we know that fitted EGARCH (1, 1) model has no autocorrelation and the model contains most of the information of MSFT.

The bootstrap method is used for the forecasting. Figure 3 shows the bootstrap forecast error bands of EGARCH (1, 1) model of returns and sigma. The forecast log-returns and sigma are shown in red line and the minimum value, 25%, 50%, 75% quantiles are shown in other colors.

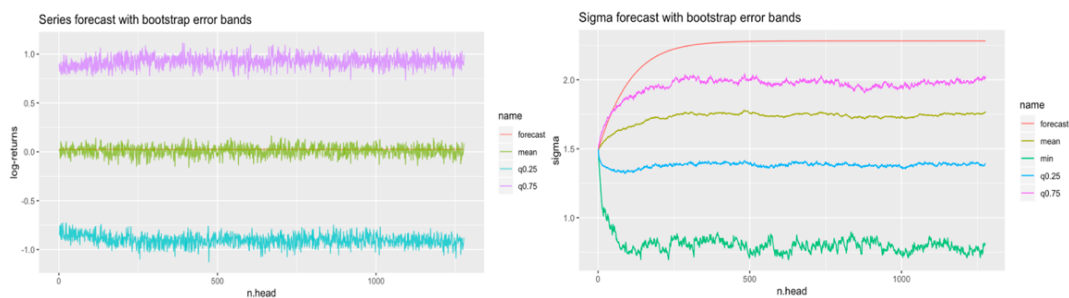


Figure 3: EGARCH (1, 1) bootstrap forecast results.

4.4. Comparison of Three Models

To consider how well the three fitted models perform on test data, the mean absolute error (MAE), root mean squared error (RMSE), mean absolute percentage error (MAPE) and mean absolute scaled error (MASE) of residuals and forecasting errors are calculated and compared. From the results, ARSV (1) has smallest residuals, which means it can reproduce the original financial data series more properly than the other two models. The forecasting results from GARCH (1,1) have slightly smaller forecast errors than EGARCH (1, 1) and ARSV (1), but all three models have similar forecast errors.

From the discussion of three models in Section 3, we focus on describing how those models match four stylized features and get three formulas related to those features, the kurtosis, persistence and the first-order autocorrelation, which are calculated for all fitted models. From the results, the persistence of fitted ARSV (1) is smaller than that of GARCH (1, 1) and EGARCH (1, 1). The kurtosis and persistence of fitted GARCH (1, 1) and EGARCH (1, 1) are similar. It can be noticed

that the estimated properties of fitted GARCH (1, 1) are closer to that of EGARCH (1, 1). With respect to the first stylized feature, the kurtosis of fitted ARSV (1) for MSFT and EUR/USD are closer to the sample kurtosis, 12.80778 and 4.76669. While the kurtosis of GARCH (1, 1) and EGARCH (1, 1) for crude oil are closer to the sample kurtosis of 7.145502. Thus, ARSV (1) can be more appropriate to match the kurtosis of sample financial data series than GARCH type model. Finally, regarding to the third and fourth features, the value of first-order autocorrelation in ARSV (1) are smaller than in the GARCH (1, 1) and EGARCH (1, 1) in all three financial data series.

5. Conclusion

In this article, we compare GARCH, EGARCH and ARSV models that can capture four stylized features from empirical and theoretical perspective. For the empirical conclusion, the ARSV model can reproduce the original financial data series more properly and the forecasting results from three models does not generate large difference. The kurtosis, persistence and first-order autocorrelation value of the fitted GARCH model are similar to EGARCH model, but rather different from ARSV model. For the theoretical conclusion, ARSV model can be a better choice when the squared observations have high kurtosis and low first-order autocorrelation, while GARCH model can also be applied under the leptokurtic conditional distribution. Moreover, ARSV model can generate smaller value of autocorrelations with larger range of values of parameters than GARCH and EGARCH models. Finally, ARSV and GARCH models can satisfy the low autocorrelation and high persistence at the same time, and the estimation of persistence of GARCH model is usually higher than in other two models.

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Appendix

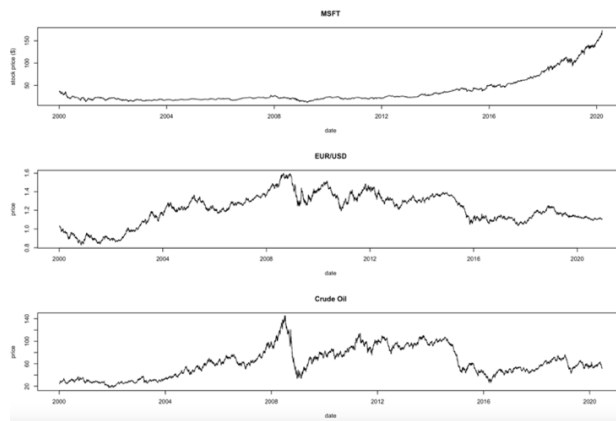
Accuracy of three fitted models on MSFT data.

GARCH (1,1)		RMSE	MAE	MAPE	MASE
	residuals	2.0215	1.3469	243.0709	0.6794
	forecast errors	1.4581	0.9944	143.0806	0.6579
EGARCH (1,1)		RMSE	MAE	MAPE	MASE
	residuals	2.0212	1.3464	180.6801	0.6791
	forecast errors	1.4644	0.9991	40.5895	0.6610
ARSV (1)		RMSE	MAE	MAPE	MASE
	residuals	0.0053	0.0039	0.0299	0.0020
	forecast errors	1.4648	0.9998	123.7492	0.6615

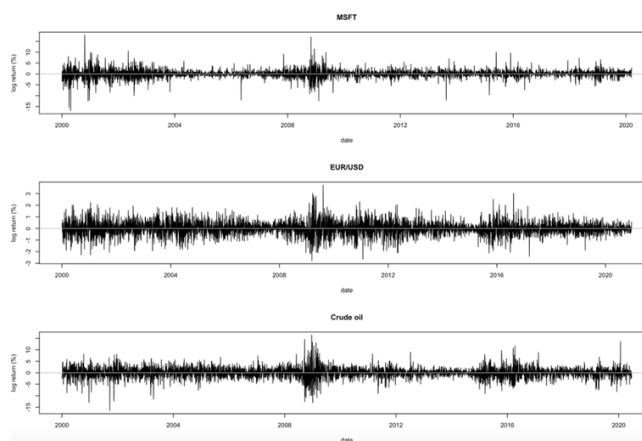
Parameters, kurtosis, persistence and first-order autocorrelation of three models.

		GARCH (1, 1)	ARSV (1)	EGARCH (1, 1)
MSFT				
		$a_0 = 0.0455$	$\mu = 0.806$	$\omega = 0.01831$
		$a_1 = 0.0623$	$\sigma_\eta^2 = 0.061$	$\alpha = -0.04361$
		$b_1 = 0.9261$	$\phi = 0.972$	$\beta = 0.9889$
				$\gamma = 0.0961$
	k_4	4.5281	9.0554	4.5433
	persistence	0.9884	0.972	0.9889
	ρ_1	0.1960	0.0279	0.14802
EUR/USD				
		$a_0 = 0.0011$	$\mu = -1.0966$	$\omega = -0.0035$
		$a_1 = 0.0334$	$\sigma_\eta^2 = 0.0042$	$\alpha = -0.0122$
		$b_1 = 0.9641$	$\phi = 0.9949$	$\beta = 0.9950$

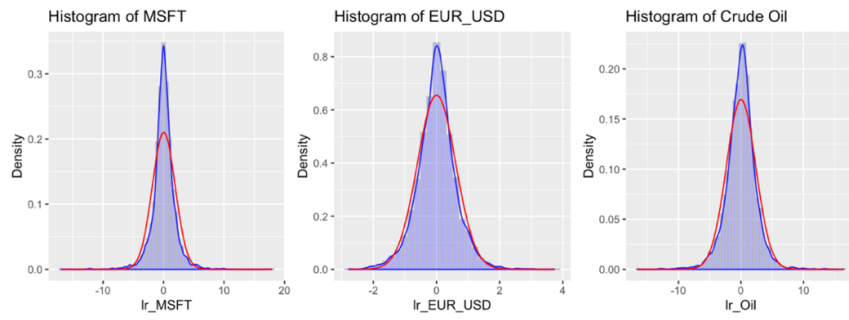
				$\gamma = 0.0716$
	k_4	5.3111	4.5332	4.9848
	persistence	0.9974	0.9949	0.9950
	ρ_1	0.2056	0.0021	0.0951
Crude Oil				
		$a_0 = 0.0233$	$\mu = 1.404$	$\omega = 0.0135$
		$a_1 = 0.0537$	$\sigma_\eta^2 = 0.012$	$\alpha = 0.0441$
		$b_1 = 0.9436$	$\phi = 0.990$	$\beta = 0.9929$
				$\gamma = 0.1053$
	k_4	6.5013	5.4829	6.4890
	persistence	0.9972	0.990	0.9929
	ρ_1	0.3777	0.0059	0.1671952



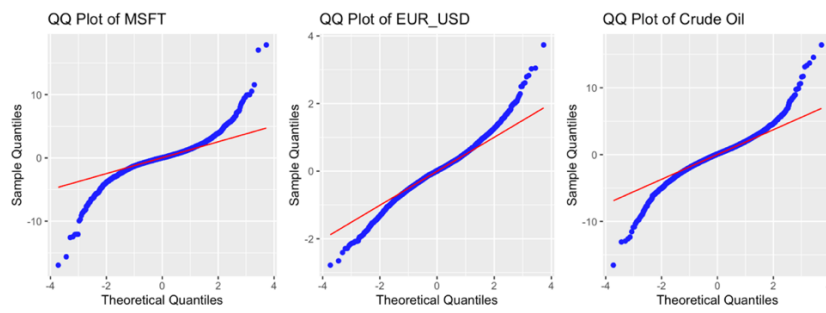
Daily closing price of three financial assets from January 1, 2000 to January 31, 2020.



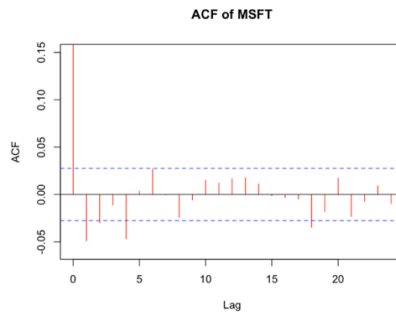
Daily log returns of three financial assets from January 1, 2000 to January 31, 2020.



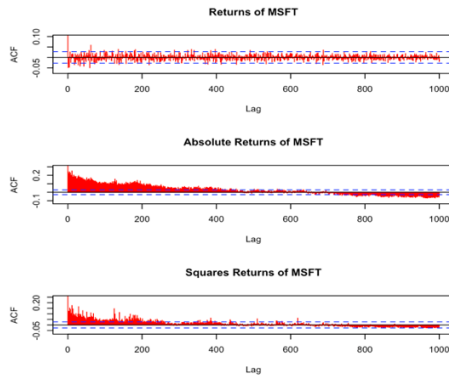
Histogram of three financial assets against theoretical normal distribution (red line).



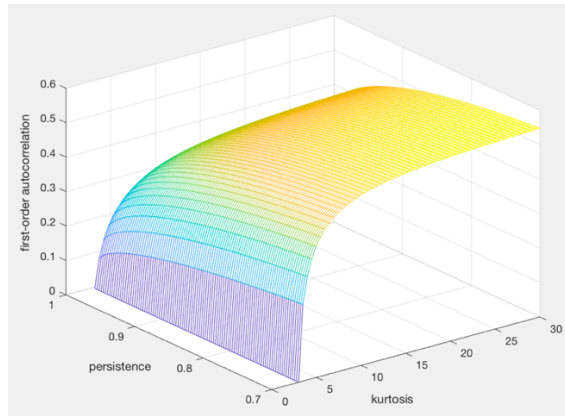
QQ plot of three financial assets against theoretical normal distribution.



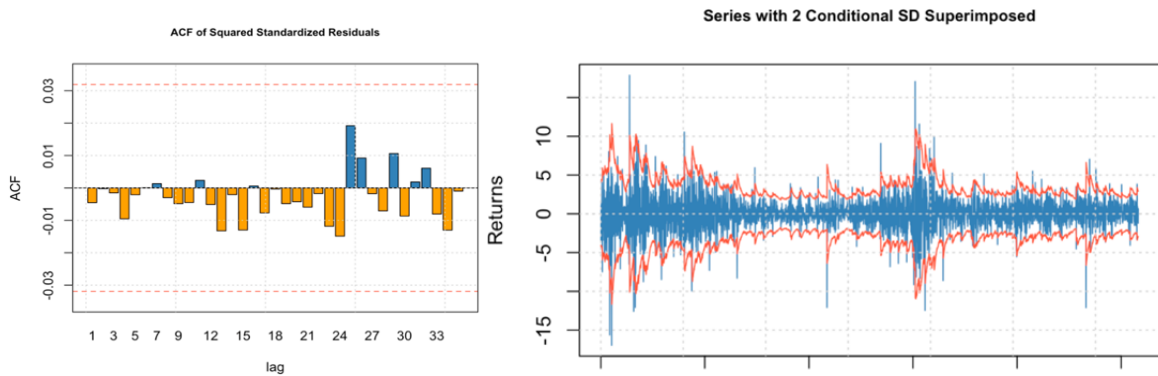
Autocorrelation plot of MSFT with 95% confidence interval for first 24 lags.



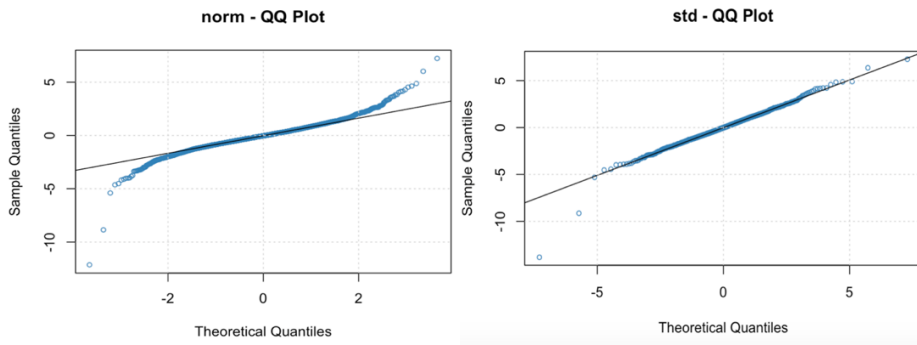
Autocorrelation plot of log-returns(top), absolute log-returns(middle) and squared log-returns(bottom) of MSFT for first 1000 lags.



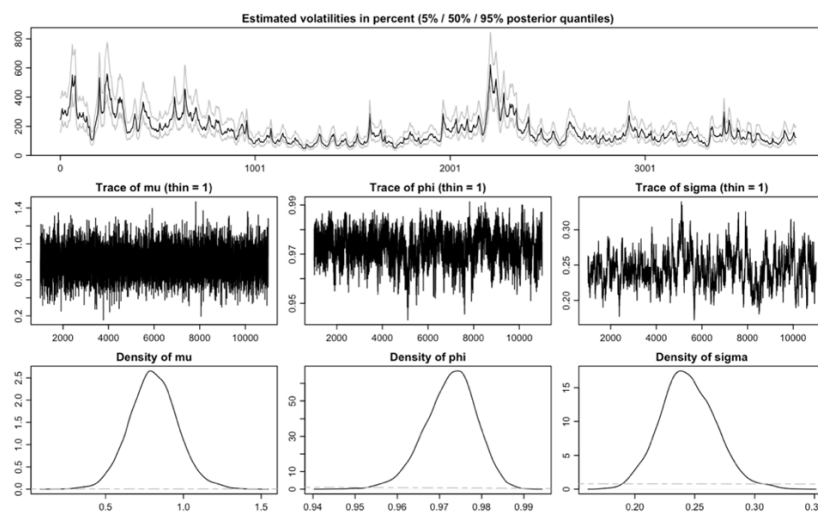
Relationship between kurtosis, persistence and first-order autocorrelation of GARCH (1, 1) with normal errors.



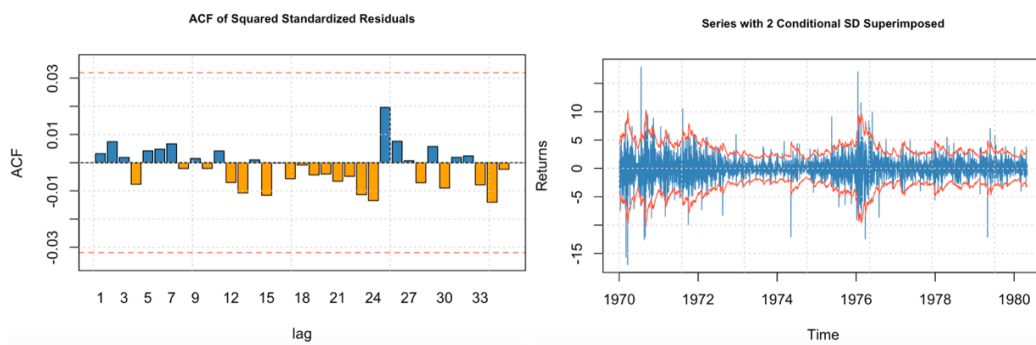
ACF and Fitted series with two conditional SD superimposed of GARCH (1, 1) for MSFT.



QQ plot of standard GARCH (left) and of GARCH-t (right) for MSFT.



Default plot of ARSV (1) model for MSFT.



ACF and Fitted series with two conditional SD superimposed of EGARCH (1, 1) for MSFT.